

Algebra of Principal Fibre Bundles, and Connections.

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In this paper, I intend to put together some of the efforts by several people of making aspects of fibre bundle theory into algebra. The initiator of these efforts was Charles Ehresmann, who put the notion of *groupoid*, and *groupoid action* in the focus for fibre bundle theory in general, and for connection theory in particular.

In so far as connection theory is concerned, this paper is a sequel to [12], and we presuppose some of the notions presented there: those of Sections 1, 3, 7, 8, and 11, so they will be recalled only sketchily. (The paper may also partly be seen as a rewriting of [7].)

1 Principal Fibre Bundles

Let us consider a groupoid object $\Phi \rightrightarrows C$ in a left exact category \underline{E} . Let us also consider a subobject $A \subseteq C$ and a global section $* : 1 \rightarrow C$. We shall talk about \underline{E} as if it were the category of sets, so we may say “subset” instead of “subobject”. We assume that the domain- and codomain formation maps are effective descent maps in \underline{E} , and that the groupoid is *transitive*, meaning that “the anchor” map $\langle d_0, d_1 \rangle : \Phi \rightarrow C \times C$ is also an effective descent map.

Then the set P of those arrows of Φ , whose codomain is in A and whose domain is $*$, carries the structure of a principal fibre bundle over A , with group $G = \Phi(*, *)$. Any principal fibre bundle in \underline{E} comes about this way from a groupoid (see the remarks below). The algebraic structure of P comes about from that of Φ , and may be made explicit as follows. First, codomain formation $d_1 : \Phi \rightarrow C$ restricts to a map $\pi : P \rightarrow A$, which is the structural

map of the bundle. The group $G = \Phi(*, *)$ acts from the right on P , by precomposition (we compose from right to left). Clearly, this action is free, and transitive on the fibres $\pi^{-1}(a)$ ($a \in A$).

Any element g of $G = \Phi(*, *)$ may, for any $a \in A$, be written in the form $x^{-1}z$ for a pair of elements in $\pi^{-1}(a)$. This representation of elements in G by “fractions” $x^{-1}z$ prompts us to use the (Ehresmann) notation $P^{-1}P$ for G . Then clearly $x \cdot x^{-1}z = z$ (where \cdot denotes the G -action). Any choice of $x \in \pi^{-1}(a)$ provides us with an explicit bijection $\pi^{-1}(a) \rightarrow G$, given by $z \mapsto x^{-1}z$.

Let us also consider the (transitive) subgroupoid Φ_A of Φ consisting of those arrows whose domain and codomain both belong to A . The groupoid Φ_A acts on the left on $P \rightarrow A$ by postcomposition; any arrow $a \rightarrow b$ in it may be presented in the form yx^{-1} , for some $x \in \pi^{-1}(a)$ and $y \in \pi^{-1}(b)$. Then clearly $yx^{-1} \cdot x = y$ (where \cdot denotes the action). The representation of arrows in Φ_A by “fractions” yx^{-1} prompts us to use the (Ehresmann) notation PP^{-1} for Φ_A .

Remark. The set P itself carries a partially defined ternary operation, given by the composite $yx^{-1}z$ in Φ (defined subject to the book-keeping condition that $\pi(x) = \pi(z)$), and this operation satisfies a couple of equations and book-keeping conditions, making it into a “pregroupoid” on A , in the sense of [7]. Out of such pregroupoid, a transitive groupoid Φ on $A + 1$ may be constructed, which in turn gives rise to P by the procedure described above (provided $\pi : P \rightarrow A$ is an effective descent map); this is in essence demonstrated in [7]. Principal fibre bundles (in the classical sense) $P \rightarrow A$, in the category of smooth manifolds, say, may, in a rather evident way, be provided with pregroupoid structure. So our “groupoid theoretic” way of describing the notion of principal fibre bundle subsumes the classical notion, and it is essentially Ehresmann’s conception. — A (non-transitive) generalization where $* : 1 \rightarrow C$ is replaced by a subset $B \subseteq C$, is considered in [9]; this generalization is relevant for foliation theory, cf. loc.cit. and [11].

We shall henceforth be interested in the case where the “base” A of the bundle $P \rightarrow A$ is to be thought of as a manifold, so we denote it by M rather than by A .

Remark on fibre bundles in general. A principal fibre bundle $P \rightarrow M$ with group G , may by the above be identified with a groupoid Φ with set of objects $M + 1$, (and with $G = \Phi(*, *)$, where $*$ is the isolated point of $M+1$). Similarly, a fibre bundle $\pi : E \rightarrow M$, with associated principal bundle

P and with fibre a left G -set F , becomes identified with a discrete opfibration over Φ (in the algebraic sense, i.e. an *action* by Φ), with $F = \pi^{-1}(*)$, and $E = \pi^{-1}(M)$. Such fibre bundle is determined up to isomorphism by P and F (with its left G -action). In the present general context, this is the upshot of [8]. We shall not explicitly be using this correspondence for general fibre bundles here. But let us remark that P itself is a fibre bundle, with fibre G (with G -action by left multiplication). Likewise, if we let G act on G by conjugation, $g \cdot h := ghg^{-1}$, we get a group bundle, namely what [14] calls the *gauge group bundle* of PP^{-1} . (It is also known under the name $Ad(P)$.) We shall utilize this latter bundle, but shall recall it without reference to this general fibre bundle theory. For a groupoid $\Psi \rightrightarrows M$, the gauge group bundle **gauge**(Ψ) is a bundle over M , which for its fibre over $a \in M$ simply has the group

$$(\mathbf{gauge}(\Psi))_a = \Psi(a, a).$$

It carries a left action by Ψ , given by conjugation: if $f : a \rightarrow b$ in Ψ and $h \in \Psi(a, a)$, then $fhf^{-1} \in \Psi(b, b)$.

The existence, for any principal fibre bundle P , of an embedding of P into a groupoid Φ , implies a “metatheorem”, namely that we may calculate freely with expressions, like vu^{-1} , as if we were dealing with actual compositions in a groupoid. The ‘action’ dots, like in $yx^{-1} \cdot x$ are then superfluous, and the same applies to many parentheses; so they are mainly kept for readability. The message (which I also tried to get through in [12] and in several other places) is that a fair amount of calculations in geometry can be performed on this very basic “multiplicative” level.

Since an arrow $f : a \rightarrow b$ in the groupoid PP^{-1} may be represented as a “fraction” yx^{-1} (with $y \in P_b$ and $x \in P_a$), it follows that an element h over a in the gauge group bundle $\mathbf{gauge}(PP^{-1})$ may be represented by a fraction yx^{-1} with y and x both $\in P_a$. For the case where the group G is *commutative*, it is well known, and easy to see, that we have an isomorphism of group bundles

$$\mathbf{gauge}(PP^{-1}) \cong M \times G, \tag{1}$$

given by sending $h = yx^{-1} \in PP^{-1}$ to $x^{-1}y \in P^{-1}P$. This cannot be done for non-commutative G : for any $g \in G$, the same h may also be represented by the fraction $yg(xg)^{-1}$, but $(xg)^{-1}(yg) = g^{-1}(x^{-1}y)g$ which is not equal to $x^{-1}y$ in general.

2 Connections versus connection forms

Consider a principal bundle $\pi : P \rightarrow M$, with group G , as above. We shall assume that M and P are equipped with reflexive symmetric relations \sim , called the *neighbour* relation. The set of pairs $(x, y) \in M \times M$ with $x \sim y$ is a subset $M_{(1)} \subseteq M \times M$, called the *first neighbourhood of the diagonal*, and similarly for $P_{(1)} \subseteq P \times P$. We assume that $\pi : P \rightarrow M$ preserves the relation \sim , and also that it is an “open submersion” in the sense that if $a \sim b$ in M , and $\pi(x) = a$, then there exists a $y \sim x$ in P with $\pi(y) = b$. In fact, we assume that for any “infinitesimal k -simplex” a_0, \dots, a_k in M (meaning a $k + 1$ -tuple of mutual neighbours), and for any $x_0 \in P$ above a_0 , there exists an infinitesimal k -simplex x_0, \dots, x_k in P (with the given first vertex x_0) which by π maps to a_0, \dots, a_k . Finally, the action of any $g \in G$ on P is assumed to preserve the relation \sim on P .

This is motivated by Synthetic Differential Geometry (SDG), cf. [4], and more recently [12], where the notion of connection (infinitesimal parallel transport) and differential form is elaborated in these terms.

The groupoid viewpoint for connections is also in essence due to Ehresmann. In SDG, this connection notion becomes paraphrased (see [7], [10] or [12], Section 8): for a groupoid $\Phi \rightrightarrows M$, a connection in it is just a map $\nabla : M_{(1)} \rightarrow \Phi$ of reflexive symmetric graphs over M .

Let $\pi : P \rightarrow M$ be a principal fibre bundle. To any connection ∇ in the groupoid PP^{-1} , one may associate a 1-form ω on P with values in the group $P^{-1}P$, as follows. For u and v neighbours in P , with $\pi(u) = a$, $\pi(v) = b$, put

$$\omega(u, v) := u^{-1}(\nabla(a, b) \cdot v). \quad (2)$$

Note that both u and $\nabla(a, b) \cdot v$ are in the π -fibre over a , so that the “fraction” $u^{-1}(\nabla(a, b) \cdot v)$ makes sense as an element of $P^{-1}P$.

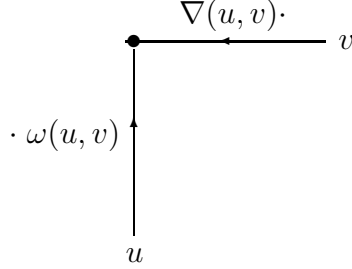
The defining equation is equivalent to

$$u \cdot \underbrace{\omega(u, v)}_{\in P^{-1}P} = \underbrace{\nabla(\pi(u), \pi(v))}_{\in PP^{-1}} \cdot v. \quad (3)$$

If we agree that (for u, v in P a pair of neighbours in P) $\nabla(u, v)$ denotes $\nabla(\pi(u), \pi(v))$, this equation may be written more succinctly

$$u \cdot \omega(u, v) = \nabla(u, v) \cdot v. \quad (4)$$

It is possible to represent the relationship between ∇ and the associated ω by means of a simple figure:



The figure reflects something geometric, namely that $\omega(u, v)$ acts inside the fibre (vertically), whereas ∇ defines a notion of horizontality.

We have the following two equations for ω . First, let $x \sim y$ in P , and assume that g has the property that also $xg \sim y$. Then

$$\omega(xg, y) = g^{-1}\omega(x, y). \quad (5)$$

Also, for $x \sim y$ and any $g \in G$

$$\omega(xg, yg) = g^{-1}\omega(x, y)g. \quad (6)$$

To prove (5), let us denote $\pi(x) = \pi(xg)$ by a and $\pi(y)$ by b . Then we have, using the defining equation (3) for ω twice,

$$xg \omega(xg, y) = \nabla(a, b)y = x\omega(x, y),$$

and now we may calculate as in a groupoid: first cancel the x on the left, then multiply the equation by g^{-1} on the left. To prove (6), we have, with a and b as above,

$$xg \omega(xg, yg) = \nabla(a, b)yg = x\omega(x, y)g,$$

by the defining equation (3) for $\omega(xg, yg)$, and by (3) for $\omega(x, y)$, multiplied on the right by g , respectively. From this, we get the result by first cancelling x and then multiplying the equation by g^{-1} on the left.

The following Proposition is now the rendering, in our context, of the relationship between a connection ∇ and its connection 1-form ω :

Proposition 1 *The process $\nabla \mapsto \omega$ just described, establishes a bijective correspondence between 1-forms ω on P , with values in the group $P^{-1}P$ and satisfying (5) and (6), and connections ∇ in the groupoid PP^{-1} .*

Proof. Given a 1-form ω satisfying (5) and (6), we construct a connection ∇ as follows. Let $a \sim b$ in M . To define the arrow $\nabla(a, b)$ in PP^{-1} , pick $u \sim v$ above $a \sim b$, and put

$$\nabla(a, b) = u(v\omega(v, u))^{-1}.$$

We first argue that this is independent of the choice of v , once u is chosen. Replacing v by $vg \sim u$, we are in the situation where (5) may be applied; we get

$$u(vg \omega(vg, u))^{-1} = u(vg g^{-1} \omega(v, u))^{-1} = u(v\omega(v, u))^{-1};$$

the left hand side is $\nabla(a, b)$ defined using u, vg , the right hand side is using u, v .

To prove independence of choice of u : any other choice is of form ug for some $g \in G$. For our new v , we now chose vg (the result will not depend on the choice, by the argument just given). Again we calculate. By (6), we have the first equality sign in

$$ug(vg\omega(vg, ug))^{-1} = ug(vgg^{-1}\omega(u, v)g)^{-1} = ug(v\omega(u, v)g)^{-1} = u(v\omega(u, v))^{-1},$$

and the two expressions here are $\nabla(a, b)$ defined using, respectively, ug, vg and u, v .

The calculation that the two processes are inverse of each other is trivial (using $\omega(u, v) = \omega(v, u)^{-1}$ and $\nabla(a, b) = \nabla(b, a)^{-1}$).

3 Gauge forms versus horizontal equivariant forms

We consider a principal fibre bundle $\pi : P \rightarrow M$ as in the previous section. The *horizontal* k -forms that we now consider, are k -forms on P with values in the group $G = P^{-1}P$. *Horizontality* means for a k -form θ that

$$\theta(u_0, u_1, \dots, u_k) = \theta(u_0, u_1 \cdot g_1, \dots, u_k \cdot g_k) \quad (7)$$

for any infinitesimal k -simplex (u_0, u_1, \dots, u_k) in P , and any $g_1, \dots, g_k \in P^{-1}P$ with the property that $(u_0, u_1 \cdot g_1, \dots, u_k \cdot g_k)$ is still an infinitesimal simplex (which is a strong "smallness" requirement on the g_i 's).

Note that the connection form ω for a connection ∇ is *not* a horizontal 1-form, since $\omega(x, yg) = \omega(x, y)g$, not $= \omega(x, y)$.

We say that a k form θ , as above, is *equivariant* if for any infinitesimal k -simplex (u_0, \dots, u_k) , and any $g \in P^{-1}P$, we have

$$\theta(u_0 \cdot g, u_1 \cdot g, \dots, u_k \cdot g) = g^{-1} \theta(u_0, u_1, \dots, u_k) g. \quad (8)$$

Note that connection forms are equivariant in this sense, by (6).

Proposition 2 *Assume that the group $G = P^{-1}P$ is commutative. Then any horizontal equivariant k -form θ on P can be written $\pi^*(\Theta)$ for a unique G -valued k -form Θ on the base space M .*

Proof. It is evident that any form $\pi^*(\Theta)$ is horizontal and equivariant (which here is better called *invariant*, since the equivariance condition now reads $\theta(u_0 \cdot g, u_1 \cdot g, \dots, u_k \cdot g) = \theta(u_0, u_1, \dots, u_k)$). Conversely, given an equivariant (= invariant) k -form θ on P , and given an infinitesimal k -simplex a_0, \dots, a_k in M , define

$$\Theta(a_0, \dots, a_k) := \theta(x_0, \dots, x_k)$$

where x_0, \dots, x_k is any infinitesimal k -simplex above a_0, \dots, a_k . The proof that this value does not depend on the choice of the x_i 's proceeds much like the proof of the well-definedness of a connection given a connection-form, in Proposition 1 above: First we prove, for fixed x_0 above a_0 , that the value is independent of the choice of the remaining x_i 's, and this is clear from the verticality assumption on θ . Next we prove that changing x_0 to $x_0 \cdot g$ (and picking $x_1 \cdot g, \dots, x_k \cdot g$ for the remaining vertices in the new k -simplex) does not change the value either, and this is clear from equivariance (= invariance).

Recall that a k -form with values in a group bundle $E \rightarrow M$ associates to an infinitesimal k -simplex a_0, \dots, a_k in M an element in the fibre of E_{a_0} . We are interested in the case where E is the gauge group bundle of a groupoid; such forms we call *gauge forms*, for brevity.

Proposition 3 *There is a natural bijective correspondence between horizontal equivariant k -forms on P with values in $G = P^{-1}P$, and k -forms on M with values in the gauge group bundle $\mathbf{gauge}(PP^{-1})$.*

Proof/Construction. Given a horizontal equivariant k -form θ on P as above, we construct a gauge valued k -form $\check{\theta}$ on M by the formula

$$\check{\theta}(a_0, \dots, a_k) := (u_0 \cdot \theta(u_0, \dots, u_k)) u_0^{-1}, \quad (9)$$

where (u_0, \dots, u_k) is an arbitrary infinitesimal k -simplex mapping to the infinitesimal k -simplex (a_0, \dots, a_k) by π (such exist, since π is a surjective submersion). Note that the numerator and the denominator in the fraction defining the value of $\check{\theta}$ are both in the fibre over x_0 , so that the value is an endo-map at a_0 in the groupoid PP^{-1} , thus does belong to the gauge group bundle. — We need to argue that this value does not depend on the choice of the infinitesimal simplex (u_0, \dots, u_k) . We first argue that, once u_0 is chosen, the choice of the remaining u_i 's in their respective fibres does not change the value. This follows from (7). To see that the value does not depend on the choice of u_0 : choosing another one amounts to choosing some $u_0 \cdot g$, for some g . But then we just change u_1, \dots, u_k by the same g ; this will give the arrow in PP^{-1}

$$(u_0 \cdot g \cdot \theta(u_0 \cdot g, \dots, u_k \cdot g))(u_0 \cdot g)^{-1}.$$

Now we calculate using the “metatheorem”, so we drop parentheses and multiplication dots; using the assumed equivariance (8), this expression then yields

$$u_0 g g^{-1} \theta(u_0, \dots, u_k) g g^{-1} u_0^{-1},$$

which clearly equals the expression in (9).

Conversely, given a gauge valued k -form α on M , we construct a $P^{-1}P$ -valued k -form $\hat{\alpha}$ on P by putting

$$\hat{\alpha}(u_0, u_1, \dots, u_k) := u_0^{-1}(\alpha(a_0, a_1, \dots, a_k) \cdot u_0) \quad (10)$$

where a_i denotes $\pi(u_i)$. Since, for $i \geq 1$, this expression depends on u_i only through $\pi(u_i) = a_i$, it is clear that (7) holds, so the form $\hat{\alpha}$ is horizontal. Also,

$$\hat{\alpha}(u_0 \cdot g, \dots, u_k \cdot g) = (u_0 \cdot g)^{-1}(\alpha(a_0, \dots, a_k) \cdot (u_0 \cdot g));$$

by the metatheorem, this immediately calculates to the expression in (10).

Finally, a calculation with the metatheorem again (cancelling u_0^{-1} with u_0) immediately gives that the two processes $\theta \mapsto \check{\theta}$ and $\alpha \mapsto \hat{\alpha}$ are inverse to each other.

We may summarize the bijection $\alpha \mapsto \hat{\alpha}$ from **gauge** (PP^{-1}) -valued forms on M to horizontal equivariant $P^{-1}P$ -valued forms on P by the formula

$$u_0 \cdot \hat{\alpha}(u_0, \dots, u_k) = (\pi^* \alpha)(u_0, \dots, u_k) \cdot u_0. \quad (11)$$

In the case that the group $G = P^{-1}P$ is commutative, we may cancel the “external” u_0 's, and get

$$\hat{\alpha}(u_0, \dots, u_k) = (\pi^* \alpha)(u_0, \dots, u_k),$$

for all infinitesimal k -simplices u_0, \dots, u_k . So under the identification of gauge forms with G -valued forms implied by (1), we have that

$$\hat{\alpha} = \pi^* \alpha. \quad (12)$$

Recall that if ∇ and ∇_1 are two connections in a groupoid $\Phi \rightrightarrows M$, we may form a 1-form $\nabla_1 \nabla^{-1}$ with values in the gauge group bundle; it is given by

$$\nabla_1 \nabla^{-1}(a, b) = \nabla_1(a, b) \cdot \nabla(b, a).$$

For the case where the groupoid is PP^{-1} , we have the following Proposition, which we shall not use in the sequel, but include for possible future reference:

Proposition 4 *Let $P \rightarrow M$ be a principal bundle, and let ∇ and ∇_1 be two connections in the groupoid PP^{-1} . Then*

$$(\nabla_1 \nabla^{-1})^\wedge = \omega_1 \cdot \omega^{-1}$$

where ω and ω_1 are the connection forms of ∇ and ∇_1 , respectively.

Proof. Let $x \sim y$, over a and $b \in M$, respectively. Then

$$\begin{aligned} (\nabla_1 \nabla^{-1})^\wedge(x, y) &= x^{-1}(\nabla_1(a, b)\nabla(b, a)x) \\ &= x^{-1}\nabla_1(a, b)y\omega(y, x) \\ &= x^{-1}x\omega_1(x, y)\omega(y, x) \\ &= \omega_1(x, y)\omega(y, x) \\ &= (\omega_1\omega^{-1})(x, y), \end{aligned}$$

using the defining relation (11) for $(-)^^\wedge$, and the relation (3) for ∇ and ∇_1 , respectively.

4 Curvature versus coboundary

Recall that the *curvature* of a connection in a groupoid $\Phi \rightrightarrows M$ is the **Gauge**(Φ)-valued 2-form $R = R_\nabla$ given by

$$R(a_0, a_1, a_2) = \nabla(a_0, a_1) \cdot \nabla(a_1, a_2) \cdot \nabla(a_2, a_0),$$

and recall that if ω is a 1-form with values in a group G , then $d\omega$ is the G -valued 2-form given by

$$d\omega(x_0, x_1, x_2) = \omega(x_0, x_1) \cdot \omega(x_1, x_2) \cdot \omega(x_2, x_0).$$

We apply this to the case where $\Phi = PP^{-1}$ and $G = P^{-1}P$, for a principal fibre bundle $\pi : P \rightarrow M$. Then the curvature R , which is a **gauge**(PP^{-1})-valued 2-form on M , gives, by Proposition 3, rise to a (horizontal and equivariant) $P^{-1}P$ -valued 2-form \hat{R} on P .

We then have the following:

Theorem 1 *Let $\pi : P \rightarrow M$ be a principal fibre bundle, and let ∇ be a connection in the groupoid PP^{-1} with connection form ω . Then we have an equality of $P^{-1}P$ -valued 2-forms on P :*

$$\hat{R} = d\omega.$$

In particular, $d\omega$ is horizontal and equivariant.

The form $\hat{R} = d\omega$ is the *curvature form* of the connection. See the remark below for comparison with the classical formulation.

Proof. Let x, y, z be an infinitesimal 2-simplex in P , and let $a = \pi(x)$, $b = \pi(y)$, and $c = \pi(z)$. We calculate the effect of the (left) action of the arrow $R(a, b, c)$ on x (note that $R(a, b, c)$ is an endo-arrow at a in the groupoid):

$$\begin{aligned} R(a, b, c) \cdot x &= \nabla(a, b) \cdot \nabla(b, c) \cdot \nabla(c, a) \cdot x \\ &= \nabla(a, b) \cdot \nabla(b, c) \cdot z \cdot \omega(z, x) \\ &= \nabla(a, b) \cdot y \cdot \omega(y, z) \cdot \omega(z, x) \\ &= x \cdot \omega(x, y) \cdot \omega(y, z) \cdot \omega(z, x) \\ &= x \cdot d\omega(x, y, z), \end{aligned}$$

using the defining equations for R and for $d\omega$ for the two outer equality signs, and using (3) three times for the middle three ones. This proves the Theorem.

Remark. By [4] I.18, or in more detail, [5]), there is a bijective correspondence between G -valued k -forms θ on a manifold P (where G is a Lie group, say $P^{-1}P$), and differential k -forms $\bar{\theta}$, in the classical sense, with values in

the Lie algebra \underline{g} of G (i.e. multilinear alternating maps $TP \times_P \dots \times_P TP \rightarrow \underline{g}$). Under this correspondence, the horizontal equivariant 2-form $d\omega$ considered in the Theorem corresponds to the classically considered "curvature 2-form" Ω on P , as in [16] II.4, [1] 5.3, or [2] V bis 4, (perhaps modulo a factor ± 2 , depending on the conventions chosen). This is not completely obvious, since Ω differs from the exterior derivative $d\bar{\omega}$ of the classical connection form $\bar{\omega}$ by a "correction term" $1/2[\bar{\omega}, \bar{\omega}]$ involving the Lie Bracket of \underline{g} ; or, alternatively, the curvature form comes about by modifying $d\bar{\omega}$ by a "horizontalization operator" (this "modification" also occurs in the treatment in [15]). The fact that this "correction term" (or the "modification") does not come up in our context can be explained by Theorem 5.4 in [5] (or see [4] Theorem 18.5); here it is proved that the formula $d\omega(x, y, z) = \omega(x, y) \cdot \omega(y, z) \cdot \omega(z, x)$ already contains this correction term, when translated into "classical" Lie algebra valued forms.

For the case where the group $P^{-1}P$ is commutative, we may use the isomorphism (1) to identify **gauge**(PP^{-1})-valued forms on M with $P^{-1}P$ -valued forms on M . Also, by Proposition 2, and the horizontality and equivariance of $d\omega$, there is a unique $P^{-1}P$ -valued 2-form $\Omega = \check{d}\omega$ on M with $\pi^*(\Omega) = d\omega$. We therefore have the following Corollary (notation as above), which is essentially what [15] call the infinitesimal version of Gauss-Bonnet Theorem (for the case where $G = SO(2)$):

Corollary 1 *Assume $P^{-1}P$ is commutative, and let the connection ∇ in PP^{-1} have connection form ω . Then the unique G -valued 2-form Ω on M with $\pi^*\Omega = d\omega$ is R_∇ .*

$$R_\nabla = \Omega.$$

Let us remark that [15] also gives a version of the Corollary for the non-commutative case, their Proposition 6.4.1; this, however, seems not correct. In this sense, our Theorem 1 is partly meant as a correction to Prop. 6.4.1, partly a "translation" of it into the pure multiplicative fibre bundle calculus, which is our main concern.

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